Economic Theory **24**, 887–906 (2004)
DOI: 10.1007/s00199-004-0485-5

**Economic Theory**


# The distribution of money and prices in an equilibrium with lotteries*

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Received: April 16, 2003; revised version: February 11, 2004

**Summary.** We construct a tractable 'fundamental' model of money with equilibrium heterogeneity in money balances and prices. We do so by considering randomized monetary trades in a standard search-theoretic model of money where agents can hold multiple units of indivisible 'tokens' and can offer lotteries on monetary transfers. By studying a simple trading pattern, we can analytically characterize the monetary distribution. Interestingly, such distributions match those observed in numerically simulated economies with fully divisible money and price heterogeneity.

**Keywords and Phrases:** Lotteries, Money distribution, Price dispersion, Search.

**JEL Classification Numbers:** D30, D83, E40.

## 1 Introduction

A classic question in monetary theory concerns the effect of money creation in economies when there is a non-degenerate distribution of money holdings (e.g. Bewley, 1983). Recent work has explored this question within the context of models based on the Shi-Trejos-Wright monetary search models where money has a 'fundamental' allocative role. Molico (1997), Deviatov and Wallace (2001), and Berentsen, Camera and Waller (2003) are such examples.

* The paper has benefitted from insightful comments of two anonymous referees, whom we thank. We also thank participants at the conference "Recent Developments in Money and Finance," held at Purdue University in May 2003, and the EPRI/University of Western Ontario Money Conference held in October 2003.

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The main difference of the approaches followed in these papers lies in how the authors set up their models in order to study the non-degenerate monetary distributions that arise. These modeling choices affect the extent and the cause of non-neutrality in the model. Molico (1997) studies a model of fully divisible money and goods using numerical methods. The key result is that lump-sum monetary injections are non-neutral due to redistributive and real balance effects. A second approach has relied on analytical methods in models with manageable–although less general–distributions of money holdings. By considering a model where agents can hold at most two indivisible tokens, Deviatov and Wallace (2001) show that money is non-neutral as it affects the quantities traded and the frequency of trading. Berentsen, Camera and Waller (2003) consider fully-divisible money and goods but focus on simple (two-point) distributions.[1] Changes in the money stock are neutral but changes in the money growth rate affect the distribution and the quantities traded.

This study complements this literature by proposing a model where we relax the indivisibility of money along two dimensions. Agents can hold multiple units of indivisible money, as in the divisible-goods framework of Camera and Corbae (1999). We augment it by allowing agents to engage in randomized monetary trades, as proposed by Berentsen, Molico and Wright (2002). The possibility to offer lotteries on money transfers further relaxes the indivisibility of money because it allows flexible monetary offers. This cures some of the inefficiencies arising from the indivisibility. We say ‘some’ because only average expenditure is affected – actual expenditure remains subject to nominal rigidities since the money is either spent or not.

We focus on an equilibrium where it is in every agent’s best interest to engage in ‘small’ nominal trades. To capture this notion we consider the following spending pattern. Buyers spend no more than one token per trade and spend it with a probability less than one. This leads to a tractable analytical characterization of the equilibrium distribution of money and prices, using three parameters: the initial supply of money, the curvature of preferences, and the agents’ storage capacity. The use of lotteries leads to analytical tractability mainly because in equilibrium traded quantities do not dependent on the initial quantity of money (they only depend on preferences), and every single-coincidence match leads to exchange.

The flexibility in monetary offers allowed by lotteries improves the efficiency of the decentralized monetary solution along the intensive and extensive margins. It expands the set of nominal offers and so it lessens bilateral trading inefficiencies (e.g. see Berentsen and Rocheteau, 2002). However, it cannot entirely eliminate them, due to equilibrium heterogeneity in valuations. Furthermore, the use of lotteries amplifies the beneficial distributional effects possible in models with multiple money inventories (e.g. see Camera, 2003). This raises the volume of trade, by lowering the fraction of agents who cannot buy or sell, and it also improves bilateral trading efficiency, by lowering the dispersion in valuations. A key result is that, within the equilibrium we study, changes in the initial money stock only affect the

[1] This is achieved by building on the degenerate distribution model of Lagos and Wright (2002), introducing additional trading periods.

extensive margin – the lotteries adjust to keep the quantities traded in each match unchanged.

The most striking result, perhaps, is that even under this simple trading pattern, the density function of money is hump-shaped, with few agents holding little or too much money. This is interesting, as this shape closely resembles that seen to arise from numerical simulations of economies with heterogeneous prices but fully divisible money (Molico, 1997).

What generates this result? The agent's equilibrium valuation of a token falls in his nominal wealth. It follows that the probability of a money transfer increases in the buyer's wealth but decreases in the seller's. Thus the poorest agents accumulate wealth easily, while the richest deplete it quickly. Once *averaged* across the entire set of traders, this spending pattern resembles that arising under fully divisible money, where the poor spend less than the rich but also earn more per trade. This leads to a density function with thin tails, and a coefficient of variation that is low, and decreases as money becomes more divisible.

## 2 Environment

The environment is as in the continuous-time model of Camera and Corbae (1999). There is a $[0, 1]$ continuum of infinite-lived agents of $J \geq 3$ specialization types, in equal proportions. Each type specializes in consumption and production of divisible nonstorable goods, where we let $X_i$ be the set of goods that agents of type $i$ consume but cannot produce. An agent suffers disutility $-q$ from production of $q > 0$ goods, and enjoys utility $u(q)$ from consumption of a quantity $q > 0$ of a desired good. We work with $u(q) = \frac{q^{1-\gamma}}{1-\gamma}, \gamma \in (0, 1)$, so that $q^* = 1$ is the quantity maximizing $u(q) - q$. The instantaneous discount rate is $r$.

Agents meet bilaterally according to a Poisson process with arrival rate $\alpha$. In a random match between agents of types $i$ and $i'$, the probability that $i$ produces a good in $X_{i'}$ and $i'$ produces a good in $X_i$ is zero, while the probability that $i$ produces a good in $X_{i'}$ but $i'$ does not produce a good in $X_i$ is $x \in (0, 1)$. Hence, $\alpha x$ is the rate at which an agent has a single coincidence match, when he meets someone who can either consume his production or produce what he likes.

Fiat money is randomly distributed initially in indivisible units that an individual can freely dispose of, or accumulate up to the bound $N \in \mathbb{N}$. We denote the initial money supply by $M \in [0, N]$ and the individual nominal balances by $n \in \mathbf{N} \equiv \{0, 1, .., N\}$. Let $m_n(t)$ be the probability that at date $t$ a randomly chosen agent has accumulated $n$ units of money, so that $\sum_{n=0}^{N} m_n(t) = 1$. In this case $\{m_0(t), ..., m_N(t)\}$ defines the distribution of money in the economy, a probability measure on $\mathbf{N}$ that must satisfy $M = \sum_{n=0}^{N} n m_n(t)$.

## 3 Symmetric stationary monetary equilibrium

We focus on equilibria where strategies and distributions are invariant functions of time, and agents in an identical state adopt identical strategies. For this reason, conjecture the existence of a distribution of money satisfying

$$\dot{m}_n(t) = 0 \, \forall n, t. \tag{1}$$

### 3.1 Terms of trade

Agents can be either buyers or sellers, depending on the realization of the match. Those without money can only be sellers, since exchange must be quid-pro-quo, those with $N$ money can only be buyers, due to the money inventory constraint. We refer to agents with large balances as being 'rich,' as opposed to those with small balances, the 'poor.'

We allow for the possibility of randomized exchange, along the lines of Berentsen, Molico and Wright (2002), as follows. Consider a single-coincidence match between a buyer with $b \in \mathbf{N}\backslash\{0\}$ money balances and a seller with $s \in \mathbf{N}\backslash\{N\}$ money balances. Let $d$ denote a positive monetary transfer from the buyer to the seller. Here $d$ must be feasible, that is the buyer cannot offer more than he has or than the seller is able to accept. Technically, $d \in D_{s,b} = \{1, 2, ..., \min\{b, N-s\}\}$. Let $q_{s,b}(d)$ denote the amount of goods requested by the buyer, given $d$, and let $\tau_{s,b}(d)$ denote the probability of transferring $d$ to the seller.[2]

The terms of trade are endogenously formed via bilateral bargaining. We use the generalized Nash protocol where $\theta \in [0, 1]$ is the buyer's bargaining power, and $1-\theta$ the seller's. For tractability, we restrict the buyer's strategy to choose a single value of $d$ first, and then to bargain with the seller over $q_{s,b}(d)$ and $\tau_{s,b}(d)$ . It follows that the terms of trade in this match will be defined by the list $\{d, q_{s,b}(d), \tau_{s,b}(d)\}$. By agreeing to this list, paired agents agree to implement the following trading plan. The seller produces $q_{s,b}(d)$ goods for the buyer and, conditional on $q_{s,b}(d)$, the buyer gives $d$ units of money to the seller with probability $\tau_{s,b}(d)$ and none otherwise. Ex-ante commitment to the trade is assumed, so ex-post renegotiation cannot occur.[3]

Let $V_n$ denote the stationary expected lifetime utility to an agent who has $n$ units of money, at some date. In a match between buyer $b$ and seller $s$, where the terms of trade are given by $\{d, q_{s,b}(d), \tau_{s,b}(d)\}$, the seller's expected net surplus from trade is

$$-q_{s,b}(d) + \tau_{s,b}(d) \left(V_{s+d} - V_s\right).$$

[2] A referee suggests an interesting extension would be to lift the restriction to choosing one single $d$ first, thus generalizing the model to one where $\tau_{s,b}(d)$ is a probability measure on $D_{s,b} \cup \{0\}$. This more general formulation would allow to consider strategies where buyers put probability mass on *several* possible transfers (e.g. $\tau_{s,b}(d) > 0$ for $d = 0, 1, 2, 3$), or make transfers with a deterministic component (e.g. $\tau_{s,b}(0) = 0$ and $\tau_{s,b}(d) > 0$ for $d = 1, 2, 3$). We surmise this formulation would lead to the following result. In equilibrium a buyer would always put some probability mass on the largest feasible transfer when he is in a match with a seller who values money more than the buyer, and would never do so otherwise.

[3] A referee suggests to think of this as a multi-stage process. First, the buyer chooses one $d$, and then the traders bargain over $q$ and $\tau$. Next, the seller produces the agreed-upon goods for the buyer. Finally, the lottery is run and the buyer gives $d$ units of money to the seller based on the lottery's realization. Goods transfers are deterministic, in equilibrium, as goods are divisible. Money transfers can be probabilistic due to indivisibilities (see Berentsen, Molico and Wright, 2002).

It has two components. The first is deterministic and it comprises the production loss $-q_{s,b}(d)$. The remaining component is the expected net continuation value $\tau_{s,b}(d)\,(V_{s+d} - V_s)$ from receiving $d$ units of money with probability $\tau_{s,b}(d)$. This is the continuation value $V_{s+d}$ minus the reservation value $V_s$. Similarly, the buyer's expected surplus is

$$u[q_{s,b}(d)] - \tau_{s,b}(d)\,(V_b - V_{b-d})\,.$$

Because we are interested in an economy where agents want to engage in 'small' nominal trades, we conjecture existence of an equilibrium in which every single-coincidence match sees the probabilistic exchange of exactly one unit of money. Technically, in all single-coincidence matches $(s,b)$, we have $d = 1$ and $\tau_{s,b}\,(1) \in (0,1)$, so that we drop the index $d$ when understood.

This conjectured pattern of exchange can be a monetary equilibrium only if $\forall n$

$$V_{n+1} > V_n \geq 0 \tag{2}$$

otherwise no-one would produce for money. Suppose (2) holds. When $q_{s,b} > 0$ we define the *nominal price* in the match by $\tau_{s,b} q_{s,b}^{-1}$, and the realized nominal payment is either zero or $q_{s,b}^{-1}$. Thus, randomized exchange convexifies the space of possible nominal offers, although it does not expand the set of feasible monetary transfers. In equilibrium, if a monetary transfer occurs, its amount $d = 1$ is independent of the match's composition, and the bargained nominal price.

Solving the bargaining problem, under this trading pattern, leads to the following

**Lemma 1** *Given (2), if $d = 1$ and $\tau_{s,b} \in (0,1)$ in all single coincidence matches $(s,b)$, then*

$$\tau_{s,b} = \frac{q_{s,b}}{V_{s+1} - V_s}\,\frac{1-\theta\gamma}{1-\gamma} \;\; \textit{and} \;\; q_{s,b} = \left(\frac{V_{s+1} - V_s}{V_b - V_{b-1}}\right)^{\frac{1}{\gamma}}. \tag{3}$$

*Proof.* In Appendix.

The key result is that, despite the greater flexibility on nominal offers allowed by lotteries, the quantities traded in equilibrium are generally inefficient, $q_{s,b} \neq 1$ (where $q_{s,b} > 0$ given (2)). The intuition is this. Heterogeneity in money holdings implies that a buyer meets sellers that can be richer or poorer than him. If rich and poor agents value money differently, something we later prove to be true, then nominal prices and traded quantities will vary across matches.

Technically, the payoffs in the Nash product include period utilities, but also the traders' net continuation values $V_{s+1} - V_s$ and $V_b - V_{b-1}$. These differences measure the agent's valuation of money as a function of his nominal wealth. Unless $s = b-1$, buyer and seller value money differently, thus $q^*$ cannot maximize the Nash product. If the seller values money more than the buyer, then the seller is willing to produce a lot per unit of money and the buyer wants to spend a lot. Conversely, if the buyer values money more than the seller, not only the latter wants to produce little per unit of money, but the buyer wants to moderate his expenditure. We later show that in equilibrium the value of money falls in the agent's nominal wealth, that is $\{V_{n+1} - V_n\}$ is a positive and decreasing sequence. Hence, expression (3) indicates that small purchases take place when the seller is richer than the buyer, $q_{s,b} < q^*$ if $s > b - 1$. Conversely, $q_{s,b} > q^*$ if $s < b - 1$.[4]

A second interesting result is that, given quantities and value functions, the probability of the monetary transfer falls in the buyer's bargaining power, $\tau_{s,b}$ is decreasing in $\theta$. This tells us that the nominal price of goods falls in each match as the bargaining power shifts to the buyer, a feature that we will exploit later on.

### *3.2 Value function*

Under the conjecture that $d = 1$ and $\tau_{s,b} \in (0, 1)$ $\forall b, s$, we can discuss the value function. Given the recursive structure of the problem facing an agent, the value function must satisfy

$$\rho V_0 = \sum_{b=1}^{N} m_b \left[-q_{n,b} + \tau_{n,b} \left(V_{n+1} - V_n\right)\right] \tag{4}$$

$$\begin{aligned}\rho V_n = & \sum_{s=0}^{N-1} m_s \left[u(q_{s,n}) - \tau_{s,n} \left(V_n - V_{n-1}\right)\right] \\ & + \sum_{b=1}^{N} m_b \left[-q_{n,b} + \tau_{n,b} \left(V_{n+1} - V_n\right)\right] \quad , n \neq 0, N\end{aligned} \tag{5}$$

$$\rho V_N = \sum_{s=0}^{N-1} m_s \left[u(q_{s,n}) - \tau_{s,n} \left(V_n - V_{n-1}\right)\right] \tag{6}$$

where $\rho = r/\alpha x$ captures the extent of trading frictions, acting effectively as a discount factor. Specifically, a small $\rho$ corresponds to an economy where trading opportunities arise frequently or where agents are patient. The first summation of the Bellman equation (5) indicates that the trader expects to earn surplus $u(q_{s,n}) - \tau_{s,n} \left(V_n - V_{n-1}\right)$ from matches where the agent is a buyer facing a seller with $s$ units of money. These matches occur with probability $m_s$. The agent can also earn some surplus, $-q_{n,b} + \tau_{n,b} \left(V_{n+1} - V_n\right)$, from matches where he sells to buyers holding $b$ units of money. Recall that agents without money can only be sellers, and those who have $n = N$ can only be buyers. Therefore $V_0$ is obtained by dropping the first summation from (5), and $V_N$ by dropping the second.

Using (3), equation (5) can be rearranged as

$$\rho V_n = \gamma\theta \sum_{s=0}^{N-1} m_s u\left(q_{s,n}\right) + \gamma\left(1 - \theta\right) \sum_{b=1}^{N} m_b \frac{q_{n,b}}{1 - \gamma}$$

dropping the first summation if $n = 0$, and the second if $n = N$. Notice that expected purchases, $\sum_{s=0}^{N-1} m_s u\left(q_{s,n}\right)$, and sales, $\sum_{b=1}^{N} m_b \frac{q_{n,b}}{1-\gamma}$, both contribute to the agent's lifetime utility, as every trade generates surplus to the agent. Since the surplus share is a function of the trader's bargaining power, $\theta$ and $1 - \theta$ multiply the first and second summation, respectively. The parameter $\gamma$, the inverse of

[4] This explains why randomized trades are always efficient in Berentsen, Molico and Wright (2002). They study the special case where the distribution of money is degenerate ($s = b - 1 = 0$ in all matches).

the intertemporal elasticity of substitution, appears because of the specific CRRA formulation of preferences.[5]

A definition of the monetary equilibrium, for the conjectured trading pattern, follows.

**Definition 1** *Given $N$ and $M$, a stationary monetary equilibrium with $d = 1$ and $\tau_{s,b} \in (0,1)$ $\forall s, b$ is a list $\{V_n, m_n, q_{s,b}, \tau_{s,b}\}_{n,s,b \in \mathbf{N}}$ that satisfies (1)-(6).*

## 4 Characterization of equilibrium in a special case

In proving the existence of an equilibrium where *all* monetary transfers are random, it is convenient to focus on the case $\theta = 1$. The reason is that $\tau_{s,b}$ falls in the buyer's bargaining power. Therefore an equilibrium where 'small trades' take place ($d = 1$) is easier to support when buyers can make take-it-or-leave-it offers to sellers. In this case the following holds

**Lemma 2** *Let $\theta = 1$. If $d = 1$ and $\tau_{s,b} \in (0,1)$, then $V_0 = 0$ and $V_n = a_n V_1$ for $n \geq 1$ with*

$$\rho V_1 = \frac{\gamma}{1-\gamma} \sum_{s=0}^{N-1} \frac{m_s}{a_{s+1}}, \tag{7}$$

*where $a_1 = 1$ and $\{a_n\}_{n=2}^{N}$ solves the $N-1$ recursive equations*

$$a_n^{\frac{\gamma}{1-\gamma}} (a_n - a_{n-1}) = 1. \tag{8}$$

*Moreover, $a_2 = a_2(\gamma) \in (1, 2)$ and $\{a_n - a_{n-1}\}$ is a decreasing positive sequence. Therefore, the sequence $\{V_n - V_{n-1}\}_{n=1}^{N}$ is decreasing and positive, and $0 \leq V_n < \infty$.*

*Proof.* In Appendix.

The first thing we notice is that lifetime utilities depend only on the CRRA coefficient, $\gamma$, and the distribution of money. In particular, $V_0 = 0$ when $\theta = 1$, since no surplus is ever earned from sales, and $V_n > 0$ otherwise. Furthermore, lifetime utility $V_n$ rises in money holdings, but it does so at a decreasing rate, so that in equilibrium richer agents value each unit of money increasingly less than poorer agents. Consequently, there is heterogeneity in money valuations. As we will see shortly, this has a crucial implication for the propensity to spend across matches, for the equilibrium flows of money generated by market transactions, and therefore for the distribution of money balances.

[5] Consider $u(q_{s,n}) - \tau_{s,n}(V_n - V_{n-1})$. Substitute for $\tau_{sb}$ to get $u(q_{s,b}) - \frac{q_{s,b}(V_b - V_{b-1})}{V_{s+1} - V_s} = u(q_{s,b}) \left(1 - \frac{q_{s,b} u'(q_{s,b})}{u(q_{s,b})}\right)$ once we recognize that (3) implies $u'(q_{s,b}) = \frac{V_b - V_{b-1}}{V_{s+1} - V_s}$. CRRA preferences imply $\frac{q_{s,b} u'(q_{s,b})}{u(q_{s,b})} = 1 - \gamma$.

An important result is that trade between buyer $b$ and seller $s$ takes place at a nominal price that depends entirely on the seller's nominal wealth. The nominal price of the transaction corresponds exactly to the seller's valuation of money,

$$\frac{\tau_{s,b}}{q_{s,b}} = \frac{1}{V_{s+1} - V_s}.$$

There are two implications. First, the price rises with the seller's wealth $s$, because the value of an additional unit of money, $V_{s+1} - V_s$, falls in $s$. Therefore, there is equilibrium price dispersion. Second, while an arbitrary seller $s$ sells goods at the same price $\frac{1}{V_{s+1}-V_s}$ to every buyer, the amount of goods sold and the likelihood of a money transfer hinge on the buyer's nominal wealth, $b$. Richer buyers always make larger purchases and are more likely to spend their money on average, as $q_{s,b}$ and $\tau_{s,b}$ increase in $b$.[6]

To prove it, use $V_n = a_n V_1$ and (8). Equilibrium lotteries and quantities are

$$\tau_{s,b} = \frac{a_b^{\frac{1}{1-\gamma}}}{a_{s+1} V_1} \text{ and } q_{s,b} = \left(\frac{a_b}{a_{s+1}}\right)^{\frac{1}{1-\gamma}}. \tag{9}$$

Evidently, $\{q_{s,b}\}$ and $\{\tau_{s,b}\}$ are positive sequences increasing in $b$ and decreasing in $s$. That is, (i) richer buyers buy more because they offer to spend a unit of money with a higher probability, relative to poorer buyers, and (ii) everyone buys more when they find a low price. This feature of equilibrium spending patterns is key to identifying the shape of the distribution $\{m_n\}$, as we next discuss.

### *4.1 Stationary distributions*

If $d = 1$ and $\tau_{s,b} \in (0, 1)\ \forall s, b$, then (1) gives rise to $N + 1$ steady-state conditions that, once normalized by $\alpha x$, are

$$m_1 \sum_{s=0}^{N-1} \tau_{s,1} m_s = m_0 \sum_{b=1}^{N} \tau_{0,b} m_b \tag{10}$$

$$\begin{aligned} &m_{n+1} \sum_{s=0}^{N-1} \tau_{s,n+1} m_s + m_{n-1} \sum_{b=1}^{N} \tau_{n-1,b} m_b \\ &= m_n \sum_{s=0}^{N-1} \tau_{s,n} m_s + m_n \sum_{b=1}^{N} \tau_{n,b} m_b \qquad , n \neq 0, N \end{aligned} \tag{11}$$

$$m_N \sum_{s=0}^{N-1} \tau_{s,N} m_s = m_{N-1} \sum_{b=1}^{N} \tau_{N-1,b} m_b \tag{12}$$

To interpret them, consider equation (11). Its left-hand-side collects all the inflows into $m_n$ and the right-hand-side collects the outflows. Since all trades involve (by conjecture) the stochastic exchange of only one unit of money, the endogenous variable $m_n$ grows as buyers with $n + 1$ units of money spend one unit in matches with some seller. In a steady state, the buyer transitions to a lower nominal wealth position with probability $\sum_{s=0}^{N-1} \tau_{s,b} m_s$. The second term indicates that sellers with $n - 1$ units of money can obtain one more unit with probability $\sum_{b=1}^{N} \tau_{n-1,b} m_b$.

[6] This differs in an important way from the equilibrium $d = 1$ in Camera and Corbae (1999). There, the price in the match $(b, s)$ is also $(V_{s+1} - V_s)^{-1}$. However, every buyer makes the same nominal offer to seller $s$, and so every buyer purchases an identical quantity from that seller, independent of the buyer's nominal wealth.

Outflows are due to sellers with $n$ units of money that acquire one more unit, $m_n \sum_{s=0}^{N-1} \tau_{s,n} m_s$, and buyers with $n$ units who spend one, $m_n \sum_{b=1}^{N} \tau_{n,b} m_b$. The expressions that account for changes in the extreme asset positions, 0 and $N$, are similarly explained.

If we use the equilibrium value of $\tau_{s,b}$ in (10)-(12), we obtain the following.

**Lemma 3** *Let $\theta = 1$. If $d = 1$ and $\tau_{s,b} \in (0,1)$ $\forall s, b$, then there exists a unique stationary distribution of money $\{m_n\}$ that satisfies*

$$m_n = m_0^{\frac{N-n}{N}} m_N^{\frac{n}{N}} \prod_{i=2}^{n} a_i^{\phi \frac{n-N}{N}} \prod_{j=n+1}^{N} a_j^{\phi \frac{n}{N}}, \quad n \neq 0, N \tag{13}$$

$$\sum_{n=0}^{N} m_0^{\frac{N-n}{N}} m_N^{\frac{n}{N}} = 1 \text{ and } M = \sum_{n=0}^{N} n m_n \tag{14}$$

*where $\phi = \frac{2-\gamma}{1-\gamma}$. Moreover for $n \neq 0, N$*

$$\frac{m_n^2}{m_{n+1} m_{n-1}} = \left( \frac{a_{n+1}}{a_n} \right)^{\phi}. \tag{15}$$

*Proof.* In Appendix.

Expression (15) implies that the stationary distribution has more mass in the interior, i.e. $\{m_n\}$ is a hump-shaped sequence (see Fig. 1). The reason is simple. In equilibrium $\{\tau_{s,b}\}$ is a sequence decreasing in $s$ and increasing in $b$. This means that, given $b$, poor sellers receive money more frequently than rich sellers. Thus poor sellers quickly increase their money holdings, while rich sellers do so slowly. Furthermore, given $s$, poor buyers choose to spend their money less frequently than rich buyers. Those who are poor are unlikely to get poorer and very likely to increase their wealth. The opposite is true for rich agents. Both of these features tend to generate a distribution with a large mass of agents in the center of it, thin tails, and a low coefficient of variation.

Interestingly, there is a sharp distinction between the distribution of money obtained in this study, relative to the censored-geometric distributions arising in the absence of lotteries but under a similar spending pattern.[7] The most striking feature, however, is another. The simple transaction pattern we study generates a density function remarkably similar to that numerically found when agents trade with fully divisible money under an identical bargaining protocol (Molico, 1997). This similarity emerges, despite (i) the very different underlying equilibrium spending patterns, and (ii) even when $N$ is relatively small (see Fig. 1), which limits considerably an individual's ability to spend small fractions of money balances.

The intuition is as follows. If buyers can offer any fraction of their balances there is no need to use lotteries. Poor buyers generally spend less than the rich, and poor

[7] Examples of equilibria where buyers have heterogeneous and bounded holdings, but everyone spends the same amount of money, can be found in Berentsen (2002), Camera and Corbae (1999) and Zhou (1999). One can easily verify that when $a_n = 1$ for all $n$ then (13) is as in Berentsen, or (15) is as in Zhou or Camera and Corbae.

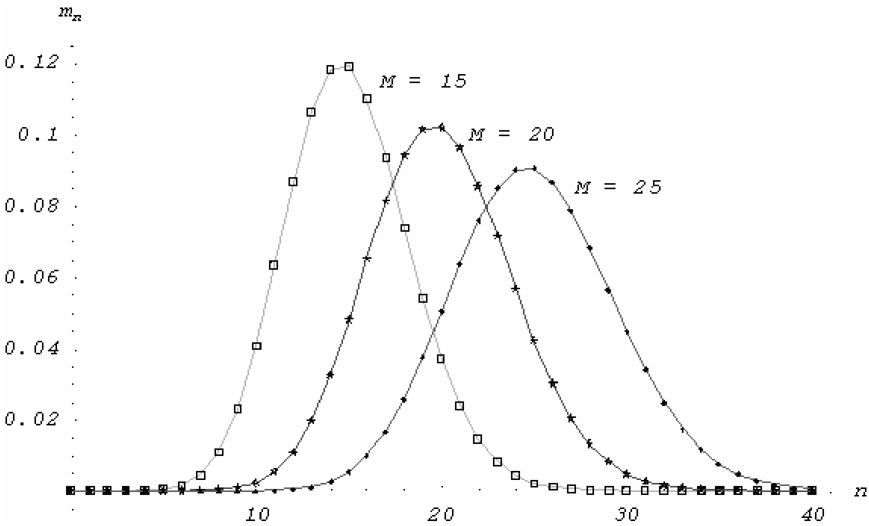


**Figure 1.** Stationary distributions for $N = 40$ and $\gamma = 0.8$, for $M = 15, 20, 25$

sellers work harder to earn more money per trade. Now consider our equilibrium with randomized trades on imperfectly divisible balances. Anyone who spends money, transfers the same amount – one unit – to every seller. The probability to make (receive) a transfer, however, increases (decreases) in the agent's wealth. Thus, our model generates monetary flows that, once averaged across the entire set of traders, resemble the monetary flows arising when nominal balances are fully divisible.

### *4.2 Individually optimal strategies*

We now provide a condition sufficient to guarantee that, under take-it-or-leave-it offers from buyers to sellers, the conjectured strategy $d = 1$ and $\tau_{s,b} \in (0, 1)$, is individually optimal in every single-coincidence match $(s, b)$. To do so, we must consider three requirements. First, given our restriction on choosing only one $d$ before bargaining over quantities and probabilities, we need to prove that no monetary transfer will involve more than one unit of money, i.e. $d = 1\ \forall s, b$. Second, we must make sure that every transfer will be *random*, i.e. $\tau_{s,b} < 1\ \forall s, b$. Finally, we need to prove that every buyer offers to spend *something* in every single-coincidence match, i.e. $\tau_{s,b} > 0\ \forall s, b$. The next lemma provides a sufficient condition capable to satisfy these three requirements.

**Lemma 4** *Let $\theta = 1$ and consider an equilibrium where $d = 1$ and $\tau_{s,b} \in (0, 1)$ in each single-coincidence match $(s, b)$. If $\rho \leq \bar{\rho}$ then this strategy is individually optimal, with*

$$\bar{\rho} = \frac{\gamma}{1-\gamma} \frac{1}{a_N^{\frac{1}{\gamma}}} \sum_{s=0}^{N-1} \frac{m_s}{a_{s+1}}. \tag{16}$$

**Table 1.** The upper bound $\bar{\rho}$

| | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 0.8$ |
|---|---|---|---|
| $N = 2$ | 0.002996 | 0.112579 | 0.98720 |
| $N = 3$ | 0.000902 | 0.12068 | 1.55035 |
| $N = 4$ | 0.000286 | 0.093182 | 1.53607 |
| $N = 5$ | 0.000112 | 0.072671 | 1.44424 |
| $N = 6$ | 0.000052 | 0.058982 | 1.36238 |

*Proof.* In Appendix.

The first step in proving this lemma is to demonstrate that no buyer deviates from equilibrium to propose a lottery on several units of money. The reasons is that doing so can only worsen the terms of trade he faces. To see why, note that our specification of preferences implies the buyer's surplus from offering a lottery on some $d$ in order to buy $q$ goods, is $\gamma u(q)$. Thus, buyers choose $d$ to consume as much as possible. Since $V_n$ is concave, a larger $d$ *lowers* the seller's valuation of the money offered, relative to the buyer's. This reduces the seller's willingness to produce per unit of money, which is bad for the buyer. If the buyer wants to consume more he should simply raise $\tau_{s,b}$, avoiding the unfavorable distortions generated by offering lotteries on larger monetary transfers.

Given that $d = 1$ is individually optimal, the next step requires us to show that in equilibrium a buyer would never offer to spend money with certainty, i.e. $\tau_{s,b} < 1$. As expected, patience is the key ingredient to achieve this. The inequality $\rho \leq \bar{\rho}$ guarantees that every monetary transfer proposed in every match will be *random*. Notice that $\bar{\rho}$ depends solely on $\gamma$, $M$, and $N$,via the sequences $\{a_n\}$ and $\{m_s\}$. Numerical analysis (see Table 1, where $M = 1.5$) shows that $\bar{\rho}$ rises in $\gamma$, and tends to fall in $N$, for $N$ large.

These findings are quite intuitive. As $N$ increases the average buyer can spend a progressively smaller fraction of his money balances. Thus as $N$ rises every buyer, including the richest, will find it less compelling to resort to lotteries. At some point the richest buyer will prefer to spend *at least* one unit of money. That is, the constraint $\tau_{0,N} \leq 1$ binds as $N$ rises above a certain threshold, given $\rho$ and $\gamma$. Now recall that $\rho$ captures the extent of trading frictions, and the curvature of preferences grows with $\gamma$. Consider a match $(b, s) = (N, 0)$ where the buyer's incentive to spend more than one unit of money is the strongest. There is a trade-off between the diminishing marginal utility and trading frictions. When $\rho$ is small the agent does not discount much the future so he limits current expenditures to spread out consumption over time. When $\gamma$ is large agents have less of an incentive to spend a lot, because marginal utility of consumption decreases very sharply. Hence, the buyer limits his current consumption by reducing the monetary offer $d$ and the probability of spending it. Thus trading more than one unit of money is suboptimal when $\rho$ is sufficiently small and $\gamma$ is sufficiently large. Note that $\tau_{0,N}$ falls as $\gamma$ rises.[8]

Finally, it is easy to show that every buyer–even the poorest–offers to spend *something* in *every* single-coincidence match. The reason is he can always offer money with a small enough probability that allows him to consume a small quantity, while limiting the risk of giving away a very valuable unit of money. Interestingly, this is quite different from models without lotteries. In those models some trades may not take place in equilibrium, when the seller values money very little, relative to the buyer, as the seller's (nominal) reservation price, $\frac{1}{V_{s+1}-V_s}$, is too high for the buyer. When lotteries on money transfers are possible, instead, the buyer can always choose a small enough probability $\tau_{s,b}$ that matches the seller's reservation price. This allows the buyer to get *at least some* consumption that generates flow utility larger than the expected loss (in terms of net continuation payoffs).

Existence of an equilibrium follows from the results listed in the previous lemmas

**Proposition 5** *Let $\theta = 1$. If $\rho \leq \bar{\rho}$, then there exists a stationary monetary equilibrium with $d = 1$ and $\tau_{s,b} \in (0, 1)$ $\forall s, b$.*

We emphasize that the allocation achieved in this equilibrium is superior to that achieved in the absence of lotteries, for two distinct reasons.

First, lotteries improve bilateral trading efficiency as agents can make nominal offers that, *on average*, are smaller than otherwise possible (see Berentsen and Rocheteau, 2002). This helps push $q_{s,b}$ closer to $q^*$ in every match, a positive ‘intensive margin’ effect. Bilateral trading inefficiencies remain, however, due to equilibrium heterogeneity in money holdings and valuations.

Second, lotteries amplify the positive ‘extensive margin’ effects associated to the agents' ability to spend only part of their balances (see Camera, 2003). The randomized money transfers foster a redistribution of money from rich to poor agents, shifting the distribution's mass closer to mean holdings and away from the tails. This has two beneficial consequences. It raises the volume of trade, by lowering the fraction of penniless agents (who cannot buy) and richest agents (who cannot sell). It also increases bilateral trading efficiency, by reducing the dispersion in money holdings, hence the disparities in valuations responsible for the inefficient selection of $q_{s,b}$.

These considerations lead us to wonder whether there is an optimum quantity of money, capable of maximizing these beneficial effects.

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[8] Notice, therefore, that the use of lotteries allows us to study economies where $N \to \infty$. Without lotteries, this is not possible since poor buyers would not buy from sellers that are too rich (see Camera and Corbae, 1999).

**Table 2.** The optimal $M$

| | $\gamma = 0.2$ | $\gamma = 0.5$ | $\gamma = 0.8$ |
|---|---|---|---|
| $N = 2$ | 1 | 1 | 1 |
| $N = 3$ | 1.4987 | 1.4921 | 1.4649 |
| $N = 4$ | 2.0168 | 1.9974 | 1.9302 |
| $N = 5$ | 2.5605 | 2.5224 | 2.4084 |
| $N = 6$ | 3.1272 | 3.0658 | 2.9016 |

### *4.3 The optimum quantity of money*

Define welfare $W$, as satisfying $W = \sum_{n=1}^{N} m_n \rho V_n$. Using $V_n = a_n V_1$ and $V_1$ from Lemma 2

$$W = \sum_{n=1}^{N} m_n a_n \rho V_1.$$

It is obvious that $W$ is a function of $M$ – since it affects the distribution of money – and of $\gamma$, that affects $\{a_n\}$ . Therefore, let $M_N^*$ denote the initial quantity of money that maximizes $W$.

For $N = 2$ one can prove that $M_2^* = N/2 = 1$ and, surprisingly, is independent of $\gamma$. In order to find $M_N^*$ for $N > 2$ we have to resort to numerical simulations (see Table 2).

The simulations suggest that the optimal quantity of money $M_N^*$ is approximately equal to $N/2$. The latter is the optimal quantity in a similar model where prices are exogenously fixed and lotteries are not allowed (see Berentsen, 2002).

The implication of this numerical experiment is that changes in the initial money stock, such that the conjectured equilibrium does not break down, are non-neutral. For $M < M^*$ there are too many agents with insufficient money balances (too few buyers) while for $M > M^*$ agents have too much money (too few sellers). Note that for given values of $N$ and $\gamma$, small changes in $M$ do not affect the quantities traded in any match since from (9) they only depend on $\gamma$. Therefore, changes in $M$ only affect the volume of trade via its effects on the extensive margin, i.e. via the distribution of money holdings.

Because non-neutralities in this model depend on the measure of poor agents, who face the most stringent constraints in their consumption ability, it is natural to ask how the distribution changes as we increase the degree of divisibility of money. More concentrated distributions would imply less significant extensive margin effects from changes in money. Below, we report the coefficient of variation as we change the degree of divisibility of money. This is done by increasing proportionately $M$ and $N$, maintaining their ratio fixed. This is equivalent to making the initial money supply more divisible, while keeping it constant (see Camera, 2003). Table 3 reports the coefficient of variation when $M/N = 0.5$ and $\gamma = 0.8$.

As money becomes more divisible we move down the coeff. of variation column, and the distribution becomes more tightly concentrated around the mean (the

**Table 3.** Divisibility and dispersion

| $N$ | $M$ | $coeff.of variation$ |
|---|---|---|
| 1 | 0.5 | 1 |
| 2 | 1 | 0.6800 |
| 3 | 1.5 | 0.5568 |
| 4 | 2 | 0.4940 |
| 5 | 2.5 | 0.4539 |
| 6 | 3 | 0.4246 |
| 7 | 3.5 | 0.4014 |
| 8 | 4 | 0.3823 |
| 9 | 4.5 | 0.3659 |
| 10 | 5 | 0.3517 |

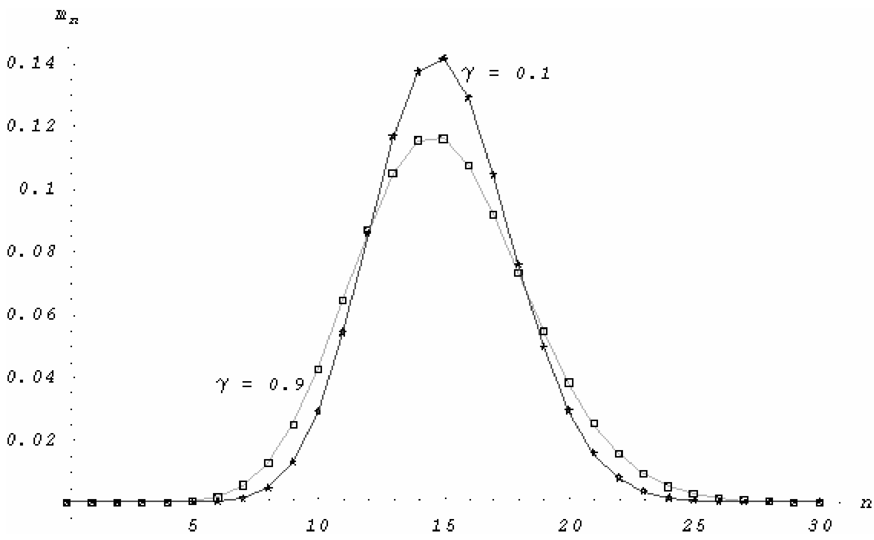


**Figure 2.** Stationary distributions for $N = 30$ amd $M = 15$, for $\gamma = 0.1, 0.9$

coefficient of variation falls). Thus, increased divisibility appears to reduce the monetary non-neutralities that impinge on a beneficial redistribution of money.

Finally, we also consider how the stationary distribution of money holdings is affected when the curvature of preferences changes. Our simulations indicate that if $\gamma$ is small, then the distribution is more concentrated around the mean. This can be explained as follows. If $\gamma$ is small, the marginal utility of consumption does not decrease very sharply, therefore agents are not so eager to smooth consumption across time. This makes them more willing to spend money to acquire goods in each meeting, which generates higher prices. Higher prices lead to a concentration of the distribution around the mean as can be seen in Figure 2.

## 5 Conclusion

We have presented an analytically tractable search-theoretic model of money that accounts for equilibrium heterogeneity in money balances and prices. The model relaxes the typical indivisibility of money of the Shi-Trejos-Wright framework by augmenting it with the possibility of holding multiple inventories of indivisible tokens and of engaging in randomized monetary trades.

The most striking result, perhaps, is the model's ability to generate monetary distributions that closely resemble those observed in numerical simulations of economies with fully divisible money and goods, and non-degenerate money distributions (Molico, 1997). The flexibility in monetary offers granted by lotteries improves the efficiency of the decentralized monetary solution along the extensive margin. It also lessens intensive margin inefficiencies, without completely curing them, however. In fact, trades remain generally inefficient since the non-degenerate equilibrium monetary distribution leads to heterogeneity in valuations of money. Because price-formation occurs via a process of bilateral bargaining, trades are inefficient when buyer and seller value differently the monetary offer. Numerical experiments indicate that as money becomes more divisible these inefficiencies are diminished since the distribution of money becomes more concentrated around the mean.

We think our approach can be successfully employed to study of a variety of issues pertinent to economies that allow for non-degeneracy of price and money holdings distributions. Such issues include the effects of money creation on welfare, and on the distribution of prices and money.

## 6 Appendix

*Proof of Lemma 1.* Suppose it is optimal for every buyer to choose $d = 1$. The optimal offer pair $\{q_{s,b}, \tau_{s,b}\}$ solves the Nash program

$$\max_{q_{s,b},\tau_{s,b}} \left[u\left(q_{s,b}\right) - \tau_{s,b}\left(V_b - V_{b-1}\right)\right]^{\theta}\left[-q_{s,b} + \tau_{s,b}\left(V_{s+1} - V_s\right)\right]^{1-\theta} \text{ s.t } \tau_{s,b} \leq 1$$

Suppose that $V_{s+1} > V_s$, otherwise no trade would take place. Substituting for $\tau_{s,b}$, consider the Lagrangian

$$\begin{aligned}\max_{\lambda_{s,b},q_{s,b},\tau_{s,b}} \quad & \left[u\left(q_{s,b}\right) - \tau_{s,b}\left(V_b - V_{b-1}\right)\right]^{\theta}\left[-q_{s,b} + \tau_{s,b}\left(V_{s+1} - V_s\right)\right]^{1-\theta} \\ & +\lambda_{s,b}\left(1 - \tau_{s,b}\right)\end{aligned}$$

where $\lambda_{s,b}$ is the multiplier on $\tau_{s,b} \leq 1$, independent of $d$ because in the equilibrium conjectured every buyer offers $d = 1$. The equilibrium $q_{s,b}$, $\tau_{s,b}$ and $\lambda_{s,b}$ must satisfy three sufficient and necessary first-order conditions

$$\begin{aligned}u'\left(q_{s,b}\right)\frac{\theta}{1-\theta} &= \frac{u(q_{s,b}) - \tau_{s,b}\left(V_b - V_{b-1}\right)}{-q_{s,b} + \tau_{s,b}\left(V_{s+1} - V_s\right)} \\ \frac{V_b - V_{b-1}}{V_{s+1} - V_s}\frac{\theta}{1-\theta} &= \frac{u(q_{s,b}) - \tau_{s,b}\left(V_b - V_{b-1}\right)}{-q_{s,b} + \tau_{s,b}\left(V_{s+1} - V_s\right)} - \lambda_{s,b}A \\ \lambda_{s,b}\left(1 - \tau_{s,b}\right) &= 0\end{aligned}$$

where $A = \left[\frac{-q_{s,b}+\tau_{s,b}(V_{s+1}-V_s)}{u(q_{s,b})-\tau_{s,b}(V_b-V_{b-1})}\right]^{\theta} \geq 0$. Note how $q_{s,b}$ and $\tau_{s,b}$ generally depend on both the seller's and the buyer's wealth positions, via their reservation values $V_{s+1} - V_s$ and $V_b - V_{b-1}$, and their relative bargaining powers, $\theta/(1-\theta)$. Two cases might arise, depending on whether the constraint $\tau_{s,b} \leq 1$ is binding or not.

1. If $\tau_{s,b} = 1$ then $\lambda_{s,b} > 0$. Hence $u'(q_{s,b}) = \frac{1-\theta}{\theta} \cdot \frac{u(q_{s,b})-(V_b-V_{b-1})}{-q_{s,b}+V_{s+1}-V_s}$.
2. If $\tau_{s,b} \leq 1$ then $\lambda_{s,b} = 0$. Hence $u'(q_{s,b}) = \frac{V_b-V_{b-1}}{V_{s+1}-V_s} \Rightarrow q_{s,b} = \left(\frac{V_{s+1}-V_s}{V_b-V_{b-1}}\right)^{\frac{1}{\gamma}}$.
   That is the marginal utility from consumption from spending $d = 1$, with probability $\tau_{s,b}$, must be equal to the ratios of the value of that unit of money to seller and buyer. Notice that since $\frac{V_{s+1}-V_s}{V_b-V_{b-1}} \neq 1$, in general, the quantity trade will be generally inefficient, unless buyer and seller 'swap' wealth positions (i.e. $s = b-1$).
   From the first order conditions we obtain
$$\begin{aligned}\tau_{s,b} &= (1-\theta)\frac{u(q_{s,b})}{V_b - V_{b-1}} + \theta\frac{q_{s,b}}{V_{s+1}-V_s} \\ &= \frac{q_{s,b}}{V_{s+1}-V_s}\left[(1-\theta)\frac{u(q_{s,b})}{q_{s,b}u'(q_{s,b})} + \theta\right] \\ &= \frac{q_{s,b}}{V_{s+1}-V_s}\frac{1-\theta\gamma}{1-\gamma}\end{aligned}$$
   so that we see that, given $\frac{q_{s,b}}{V_{s+1}-V_s}$, $\tau_{s,b}$ decreases in $\theta$. □

*Proof of Lemma 2.* If $d = 1$ and $\tau_{s,b} \in (0,1)$, then for $n \neq 0, N$

$$\rho V_n = \frac{\gamma}{1-\gamma}\sum_{s=0}^{N-1} m_s \left(\frac{V_{s+1}-V_s}{V_n - V_{n-1}}\right)^{\frac{1-\gamma}{\gamma}}$$

Note that

$$\rho V_n (V_n - V_{n-1})^{\frac{1-\gamma}{\gamma}} = \frac{\gamma}{1-\gamma}\sum_{s=0}^{N-1} m_s (V_{s+1}-V_s)^{\frac{1-\gamma}{\gamma}}$$

is independent of $n$. It follows that for $n \geq 2$ :

$$\frac{V_n (V_n - V_{n-1})^{\frac{1-\gamma}{\gamma}}}{V_{n-1}(V_{n-1}-V_{n-2})^{\frac{1-\gamma}{\gamma}}} = 1 \Rightarrow \frac{V_n}{V_{n-1}} = \left(\frac{V_{n-1}-V_{n-2}}{V_n - V_{n-1}}\right)^{\frac{1-\gamma}{\gamma}} \Rightarrow \frac{V_N}{V_1} = \left(\frac{V_1 - V_0}{V_N - V_{N-1}}\right)^{\frac{1-\gamma}{\gamma}}$$

because of a telescoping product.

If we let $a_1 = 1$, $a_0 = 0$, and $V_n = a_n V_1$ then $\frac{V_n}{V_{n-1}} = \frac{a_n}{a_{n-1}}$ and $V_n - V_{n-1} = (a_n - a_{n-1}) V_1$. Therefore we can find $\{a_n\}_{n=1}^N$ recursively:

$$\frac{a_2}{a_1} = \left(\frac{V_1-V_0}{V_2-V_1}\right)^{\frac{1-\gamma}{\gamma}} \Rightarrow \quad a_2^{\frac{\gamma}{1-\gamma}}(a_2 - 1) = 1 \quad (\text{since } a_1 = 1)$$
$$a_n^{\frac{\gamma}{1-\gamma}}(a_n - a_{n-1}) = 1 \quad \forall\, 2 < n \leq N$$

Thus $a_n$ is a function solely of $\gamma$, hinging on $a_2 = a(\gamma)$. It is easy to see that $a_2 > 1$ and $a_2 < 2$ because $a_2^{\frac{\gamma}{1-\gamma}} (a_2 - 1)$ increases in $a_2$ and at $a_2 = 2$ does not satisfy the equality above. Also, $\{a_n - a_{n-1}\}$ is a positive but decreasing sequence (because $a_n - a_{n-1} = 1/a_n^{\frac{\gamma}{1-\gamma}}$, $a_n - a_{n-1}$ must be decreasing in $n$). Therefore $V_n$ is an increasing function of $n$, and $\{V_n - V_{n-1}\}$ is a decreasing sequence.

Use the result that $V_n - V_{n-1} = (a_n - a_{n-1}) V_1$. Then:

$$\begin{aligned}
\rho V_1 (V_1 - V_0)^{\frac{1-\gamma}{\gamma}} &= \frac{\gamma}{1-\gamma} \sum_{s=0}^{N-1} m_s (V_{s+1} - V_s)^{\frac{1-\gamma}{\gamma}} \\
\rho V_1 (V_1)^{\frac{1-\gamma}{\gamma}} &= \frac{\gamma}{1-\gamma} \sum_{s=0}^{N-1} m_s (a_{s+1} - a_s)^{\frac{1-\gamma}{\gamma}} V_1^{\frac{1-\gamma}{\gamma}} \\
&\qquad\qquad (use V_s - V_{s-1} = (a_s - a_{s-1}) V_1) \\
\rho V_1 &= \frac{\gamma}{1-\gamma} \sum_{s=0}^{N-1} m_s (a_{s+1} - a_s)^{\frac{1-\gamma}{\gamma}} \\
\rho V_1 &= \frac{\gamma}{1-\gamma} \sum_{s=0}^{N-1} \frac{m_s}{a_{s+1}} \qquad (use\, (a_{s+1} - a_s)^{\frac{1-\gamma}{\gamma}} = a_{s+1}^{-1})
\end{aligned}$$

where we notice that $V_1 < \infty$ since $\{a_{s+1} - a_s\}$ is a converging sequence.

Using the definition of $a_n$, $\tau_{s,b} = \frac{a_b^{\frac{1}{\gamma}}}{a_{s+1} V_1}$. Hence, $\{\tau_{s,b}\}$ is a sequence increasing in $b$ and decreasing in $s$. □

*Proof of Lemma 3.* To start we notice that (10) - (12) imply

$$m_n \sum_{b=1}^{N} \tau_{n,b} m_b = m_{n+1} \sum_{s=0}^{N-1} \tau_{s,n+1} m_s \quad \forall n \neq N \tag{17}$$

which means that the expected money flow to sellers with $n$ units of money, must be equal to the expected money outflow of buyers with $n+1$ units of money. To see why this holds, start with (10), and then use it in (11) with $n = 1$. Observe that only the summations to the extreme left and extreme right of (11) are left (the inner summations cancel out). Then repeat it recursively, for each $n < N$.

Now use (17) replacing the lotteries by their expressions given in (9) to get

$$\begin{aligned}
& \tfrac{m_n}{a_{n+1}} \textstyle\sum_{b=1}^{N} a_b^{\frac{1}{1-\gamma}} m_b = m_{n+1} a_{n+1}^{\frac{1}{1-\gamma}} \textstyle\sum_{s=0}^{N-1} \tfrac{m_s}{a_{s+1}} && \forall n \neq N \\
\Rightarrow\quad & \tfrac{m_{n+1}}{m_n} a_{n+1}^{\frac{2-\gamma}{1-\gamma}} = \frac{\sum_{b=1}^{N} a_b^{\frac{1}{1-\gamma}} m_b}{\sum_{s=0}^{N-1} \frac{m_s}{a_{s+1}}} && \\
\Rightarrow\quad & \tfrac{m_n}{m_{n+1}} = \tfrac{m_0}{m_1} a_{n+1}^{\frac{2-\gamma}{1-\gamma}} && \forall n \neq N
\end{aligned} \tag{18}$$

since (10) implies $\frac{m_1}{m_0} = \frac{\sum_{b=1}^{N} a_b^{\frac{1}{1-\gamma}} m_b}{\sum_{s=0}^{N-1} \frac{m_s}{a_{s+1}}}$ after one substitutes for (9).

We can use the last line of (18) for any two adjacent $n$ and $n+1$ to obtain

$$\frac{m_n^2}{m_{n+1}m_{n-1}} = \left(\frac{a_{n+1}}{a_n}\right)^{\frac{2-\gamma}{1-\gamma}} \quad \forall n \neq 0, N \tag{19}$$

This tells us that $\left\{\frac{a_{n+1}}{a_n}\right\}_{n=1}^{N-1}$ is a decreasing sequence so that $\left\{\frac{m_n}{m_{n+1}}\frac{m_n}{m_{n-1}}\right\}_{n=1}^{N-1}$ is a decreasing sequence also. It follows that $\left\{\frac{m_n}{m_{n+1}}\right\}_{n=1}^{N-1}$ cannot be an increasing sequence, i.e. $m_n > m_{n+1}$ $\forall n \neq 0, N$ cannot be an equilibrium. Now use the last line of (18). We see that $m_0 > m_1$ is not possible (it would imply $m_n > m_{n+1}$ $\forall n \neq 0, N$). Thus $m_0 < m_1$ must hold. Since $m_n > m_{n+1}$ $\forall n \neq 0, N$ is not possible, then the only equilibrium is $m_n < m_{n+1}$ for some $1 \leq n < n^*$ and $m_n > m_{n+1}$ for $n \geq n^*$. That is, $\{m_n\}$ is hump-shaped.

Since $\frac{m_{n+1}}{m_0} = \frac{m_{n+1}}{m_n} \times \frac{m_n}{m_{n-1}} \times ... \times \frac{m_1}{m_0}$ and $\frac{m_n}{m_{n+1}} = \frac{m_0}{m_1} a_{n+1}^{\frac{2-\gamma}{1-\gamma}}$ then

$$\frac{m_{n+1}}{m_0} = \left(\frac{m_1}{m_0}\right)^{n+1} \Pi_{j=1}^{n+1} a_j^{-\frac{2-\gamma}{1-\gamma}} \quad \text{for all } \; n \neq N.$$

Let $A_{n+1} = \Pi_{j=1}^{n+1} a_j^{-\frac{2-\gamma}{1-\gamma}}$ for $n \neq N$, and notice that $A_0 = 1$. Then, the stationary distribution solves the system of $N+1$ non-linear equations in $N+1$ unknowns.:

$$\begin{aligned}
&m_{n+1} = m_0 \left(\tfrac{m_1}{m_0}\right)^{n+1} A_{n+1} && \forall n \neq 0, N \\
&m_0 + \textstyle\sum_{n=0}^{N-1} m_0 \left(\tfrac{m_1}{m_0}\right)^{n+1} A_{n+1} = 1 && \\
&\textstyle\sum_{n=0}^{N-1} (n+1)\, m_0 \left(\tfrac{m_1}{m_0}\right)^{n+1} A_{n+1} = M &&
\end{aligned}$$

These expressions can be rewritten to yield (13) and (14).

We next show uniqueness of the stationary distribution for any $N$ and money supply $M \in (0, N)$. The first thing to note is that $m_0 + \sum_{n=0}^{N-1} m_0 \left(\frac{m_1}{m_0}\right)^{n+1} A_{n+1} = 1$ implies $\frac{\partial m_1}{\partial m_0} < 0$. Thus, for any $N$ and $m_0$ there is a unique $M$ that satisfies $m_{n+1} = m_0 \left(\frac{m_1}{m_0}\right)^{n+1} A_{n+1}$ and $m_0 + \sum_{n=0}^{N-1} m_0 \left(\frac{m_1}{m_0}\right)^{n+1} A_{n+1} = 1$. Next, note that $\sum_{n=0}^{N-1} (n+1)\, m_0 \left(\frac{m_1}{m_0}\right)^{n+1} \times A_{n+1} = M$ implies that $m_0$ is monotonically decreasing in $M$ (recall that $\frac{\partial m_1}{\partial m_0} < 0$). Accordingly, for any $n$ and $M \in (0, N)$ there is a unique $\{m_n\}$ satisfying (13) and (14). □

*Proof of Lemma 4.* Let $\theta = 1$. Suppose $d = 1$ and $\tau_{s,b} \in (0, 1)$ $\forall b, s$ is an equilibrium. Consider the strategy of a representative buyer $b$ in a match with a seller $s$. To prove individual optimality of the strategy proposed we take three steps. Finally, we prove that every single coincidence match will result in a trade. That is the buyer always puts a positive probability on the transfer of $d = 1$.

1. First, we prove that *if* a buyer offers a lottery on the transfer of $d \in D_{s,b}$ units of money, then $d = 1$ is individually optimal. The proof is by means of contradiction.

Pick any feasible offer $d$ and suppose that the buyer wants to offer a lottery. Since the buyer extracts the seller's entire surplus, then the optimal transfer probability must satisfy $\tau_{s,b}(d) \in [0,1]$ and

$$\tau_{s,b}(d) = \frac{q_{s,b}(d)}{V_{s+d} - V_s}.$$

Given this probability, the buyer chooses $q_{s,b}(d)$ to maximize his surplus

$$u[q_{s,b}(d)] - \tau_{s,b}(d)(V_b - V_{b-d}) = u[q_{s,b}(d)] - q_{s,b}(d)\frac{V_b - V_{b-d}}{V_{s+d} - V_s}.$$

Since $u(q) = \frac{q^{1-\gamma}}{1-\gamma}$, then optimal consumption is

$$q_{s,b}(d) = \left(\frac{V_{s+d} - V_s}{V_b - V_{b-d}}\right)^{\frac{1}{\gamma}}. \tag{20}$$

The implication is that, given $d \in D_{s,b}$, when $\tau_{s,b}(d)$ and $q_{s,b}(d)$ are optimally chosen then the buyer's surplus is $u[q_{s,b}(d)] - q_{s,b}(d)q_{s,b}(d)^{\gamma}$, or

$$\gamma u[q_{s,b}(d)]. \tag{21}$$

Clearly the $d$ that maximizes (21) must generate the largest quantity, i.e. it must maximize $q_{s,b}(d) = \left(\frac{V_{s+d}-V_s}{V_b-V_{b-d}}\right)^{\frac{1}{\gamma}}$. It is easily proved that

$$\begin{aligned}\frac{V_b - V_{b-d}}{V_{s+d} - V_s} &= \frac{(V_b - V_{b-1}) + (V_{b-1} - V_{b-2}) + ... + (V_{b-d+1} - V_{b-d})}{(V_{s+d} - V_{s+d-1}) + (V_{s+d-1} - V_{s+d-2}) + ... + (V_{s+1} - V_s)} \\ &\geq \frac{V_b - V_{b-1}}{V_{s+1} - V_s} \qquad \forall d \geq 1\end{aligned}$$

since $V_b - V_{b-1} < V_{b-1} - V_{b-2} < .. < V_{b-d+1} - V_{b-d}$, while $V_{s+1} - V_s > V_{s+2} - V_{s+1} > ... > V_{s+d} - V_{s+d-1}$, because $\{V_{n+1} - V_n\}$ is a decreasing sequence, in equilibrium. That is, raising $d$ above one, increases the numerator and decreases the denominator of the ratio $\frac{V_b-V_{b-d}}{V_{s+d}-V_s}$. Since $\frac{V_b-V_{b-d}}{V_{s+d}-V_s} \geq \frac{V_b-V_{b-1}}{V_{s+1}-V_s}$ $\forall d \geq 1$, then it follows that setting $d \geq 2$ is worse than offering $d = 1$. Offering a lottery on $d \geq 2$ is suboptimal because, in the equilibrium conjectured, it simply reduces the quantity consumed by the buyer, hence his surplus.

2. Now we provide a condition guaranteeing that $\tau_{s,b} < 1$ is individually optimal. That is, offering $d = 1$ with certainty is suboptimal. In the conjectured equilibrium $\tau_{s,b}(1) = \tau_{s,b}$, defined by (3) for $\theta = 1$. Because $\{\tau_{s,b}\}$ is increasing in $b$ and decreasing in $s$, it follows that a sufficient condition for $\tau_{s,b}(1) < 1$ is $\tau_{0,N} < 1$. Using the results in the prior Lemmas, this amounts to the inequality $\frac{a_N^{1/\gamma}}{V_1} < 1$ that, substituting for $V_1$ can be rearranged as

$$\rho < \bar{\rho} = \frac{\gamma}{1-\gamma}\frac{1}{a_N^{\frac{1}{\gamma}}}\sum_{s=0}^{N-1}\frac{m_s}{a_{s+1}}.$$

It is seen that the sequences $\{a_n\}$ and $\{m_s\}$ only depend on $\gamma$, $M$, and $N$.

3. Finally, we prove that every buyer offers to spend *something* in *every* single-coincidence match, i.e. $\tau_{s,b} > 0\,\forall s, b$ is individually optimal. Since $\tau_{s,b} = \frac{q_{s,b}}{V_{s+1}-V_s}$ in equilibrium, the buyer's expected surplus is positive in every possible match, i.e. $u(q_{s,b}) - \tau_{s,b}\,(V_b - V_{b-1}) \equiv \gamma u(q_{s,b}) > 0\ \forall s, b$. In equilibrium $q_{s,b} = \left(\frac{V_{s+1}-V_s}{V_b-V_{b-1}}\right)^{\frac{1}{\gamma}}$ .Therefore $\tau_{s,b} > 0$. □

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